Brownian Motion: some elementary properties.

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Standard Brownian Motion (SBM):

$\{B(t) : 0 \le t < \infty\}$ on $(\Omega, \mathcal{F}, P)$ with properties a)–c):

a) $\forall \omega \in \Omega$ $B(0,\omega) = 0$

b) $\forall \omega \in \Omega$ $B(t,\omega)$ is continuous in t.

c) $0 < t < s$, then $B(t)$ and $B(s) - B(t)$ are independent normal distributions, mean 0, variance t and $s-t$ correspondingly (i.e. have distributions with density $\frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-a)^2}{\sigma}}$, $\sigma = t, s-t$).

Can be interpreted as a probability measure on $C[0,\infty)$.

Properties:

1) Existence (Wiener Thm) and Uniqueness up to a measure preserving transformation.

2) (Time-homogeneity) $B_{t+s} - B_s$, $t \ge 0$ – also SBM, independent of $(B_u)_{u \le s}$.

3) $0 < t_1 < t_2 < \dots < t_n$, then $B(t_1)$, $B(t_2) - B(t_1)$, .., $B(t_n) - B(t_{n-1})$ are independent. (since for normal distributions uncorrelated $\Leftrightarrow$ independent).

4) $-B_t$ is also SBM.

5) cB_{t/c^2} is also SBM for $c \ne 0$ – Brownian scaling.

6) $\langle B_s, B_t\rangle = E(B_s B_t) = \min(s,t)$. $(s > t \Rightarrow E(B_s B_t) = E((B_s - B_t)B_t) + E(B_t^2) = t)$.

7) $X_0 = 0$, $X_t = tB_{1/t}$ – SBM (Covariance: s, t $E(tB_{1/t}\, sB_{1/s}) = ts\,E(B_{1/t} B_{1/s}) = ts \cdot \frac{1}{s} = t$. Continuity on $(0,\infty)$ – obvious.

$t \to 0$: for rational pts, X_t distributed as B_t, so $\lim\limits_{\substack{t\to 0\\ t \in \mathbb{Q}}} X_t = \lim\limits_{\substack{t\to 0\\ t\in\mathbb{Q}}} B_t = 0$.

But X_t is continuous, so $\lim\limits_{t\to 0} X_t = 0$).

8) (Law of Large Numbers) $P(\lim \frac{B(t)}{t} = 0) = 1$ $(\Leftarrow \frac{B(t)}{t} = X(\frac{1}{t}))$.

9) (Levy's Modulus of continuity). $|B(t+\varepsilon) - B(t)| \le C(\omega)\sqrt{\delta \log\frac{1}{\delta}}$, where $C(\omega)$ is a.s. finite. In particular, $B(t)$ is α-Hölder $\forall \alpha < \frac{1}{2}$.

9') (Strong Levy's modulus of continuity).

$$P\left(\varlimsup_{\varepsilon\to 0} \sup_{\substack{0 \le t_1 < t_2 < 1\\ t_2 - t_1 < \varepsilon}} \frac{|B_{t_2} - B_{t_1}|}{\sqrt{2t\log\frac{1}{t}}} = 1\right) = 1.$$

10) (Law of iterated logarithm)

$$\varlimsup_{t\to 0} \frac{B_t}{\sqrt{2t\log\log\frac{1}{t}}} = 1 \text{ a.s.}$$

$$\varlimsup_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log\frac{1}{t}}} = 1 \text{ a.s. } (\Leftarrow X_t = tB_{1/t}).$$

$$\varlimsup_{t\to 0} \frac{B_{s+t} - B_s}{\sqrt{2t\log\log\frac{1}{t}}} = 1 \text{ a.s. for } \underline{\text{fixed}}\ s.$$

11) (Existence of Quadratic Variation)

X_t – real-valued process. We say that X_t has finite quadratic variation if $\forall t > 0$, $\forall (\Delta_n)$ – family of partitions of $[0,t]$: $0 = t_0^n < t_1^n < \dots < t_{k_n}^n = t$, with $|\Delta_n| = \max|t_{i+1} - t_i| \to 0$, we have $P\text{-}\lim \sum (X_{t_{i+1}} - X_{t_i})^2 = \langle X, X\rangle_t$.

Note that $\langle X, X\rangle_t$ is necessary an increasing function.

Thm $\langle B,B\rangle_t = t$.

Pf. We'll need

Wick's formula: Let $X_1,\dots,X_n$ be centered Gaussians. Then $E(X_1\dots X_n) = \sum\limits_{\substack{\text{perfect pairings of } \{1,\dots,n\} \\ \{(i_1,j_1)\dots(i_k,j_k)\}}} \prod E(X_{i_k}X_{j_k})$.

Thus for odd n it is always $=0$, and $E(X^4) = 3(E(X^2))^2$.

Pf of Wick.

$E(\prod e^{t_iX_i}) = E(e^{\sum t_iX_i}) \underset{\substack{\text{Gaussian} \\ E(e^{tX}) = e^{\frac{t^2\sigma^2}{2}}}}{=} \exp(\frac{1}{2}\|\sum t_iX_i\|_2^2) = \exp(\frac{1}{2}\sum t_it_jE(X_iX_j))$

Expand: $E(\prod e^{t_iX_i}) = E(\prod(1+t_iX_i+\dots)) = 1 + t_1t_2\dots t_nE(\prod\limits_{i=1}^n X_i) + \dots$

or $\exp(\frac{1}{2}\sum t_it_jE(X_iX_j)) = \exp(\sum\limits_{i<j} t_it_jE(X_iX_j) + \sum t_i^2 E(X_i^2)) =$

$1 + t_1\dots t_n \sum \prod E(X_{i_k}X_{j_k}) + \dots$ ∎

So, returning to BM, we'll prove more: L^2-convergence

$\|\sum(B_{t_{i+1}}-B_{t_i})^2 - t\|_2^2 = E\left(\left(\sum\left((B_{t_{i+1}}-B_{t_i})^2 - (t_{i+1}-t_i)\right)\right)^2\right) =$

$E\left(\sum(B_{t_{i+1}}-B_{t_i})^4 - 2\sum(B_{t_{i+1}}-B_{t_i})^2(t_{i+1}-t_i) + \sum(t_{i+1}-t_i)^2\right) =$

$3\sum(t_{i+1}-t_i)^2 - 2\sum(t_{i+1}-t_i)^2 + \sum(t_{i+1}-t_i)^2 \le 2|\Delta_n| t \to 0$ ∎

Filtrations, adapted processes, stopping times.

Def. $(\Omega,\mathcal{F})$ - measurable space. Filtration is an increasing family of sub-σ-algebras of $\mathcal{F}$. A measurable space with a filtration is called a filtered space. $(\mathcal{F}_t)_{t\in T}$, T - some index set. For us, $T = \mathbb{N}$ or $T = \mathbb{R}_+$.

Examples: 1) A b-adic filtration of $([0,1],\mathcal{B})$: $\mathcal{F}_n$ - generated by the b-adic intervals of n-th generation.

2) Brownian filtration: $\mathcal{F}_t$ is the smallest σ-algebra such that all $(B_s)_{s\le t}$ are measurable. Since the Brownian motion is defined on an abstract space, need $\mathcal{F}_\infty = \vee(\mathcal{F}_t)_{t\ge 0}$ ("$\vee$" - means the σ-algebra generated by the union). Can also define $\mathcal{F}_t^- = \vee_{s<t}\mathcal{F}_s = \mathcal{F}_t$ (by continuity), $\mathcal{F}_t^+ := \bigcap_{s>t}\mathcal{F}_s$.

For BM, we have:

Claim. $\{B(t+s)-B(s) : t\ge 0\}$ is independent of $\mathcal{F}_s^+$.

Pf. We know that $B(t+s)-B(s)$ is independent of $\mathcal{F}_s$, by Markov. Let $s_n \downarrow s$, then $B(t+s)-B(s) = \lim\limits_{n\to\infty}(B(t+s_n)-B(s_n))$ a.s., by continuity. But $B(t+s_n)-B(s_n)$ is independent of $\mathcal{F}_s^+$, then so is $B(t+s)-B(s)$. ∎

Corollary (Blumental's 0-1 law). $\forall A\in\mathcal{F}_0^+$, we have $P(A)=0$ or $P(A)=1$.

Pf. By Claim, $\forall A\in\mathcal{F}_\infty$, A is independent of $\mathcal{F}_0^+$. If $A\in\mathcal{F}_0^+$, it is independent of itself. So $P(A)=0$ or 1 ∎

Corollary (Zero-one tail law). Let $\mathcal{F}_{tail} := \bigcap_t \bigvee_{s>t}\mathcal{F}_s$. If $A\in\mathcal{F}_{tail}$, then so $P(A)=0$ or 1.

Corollary (Zero-one tail law). Let $\mathcal{F}_{tail} := \bigcap_{t\ge 0} \bigvee_{s\ge t} \mathcal{F}_s$. If $A\in\mathcal{F}_{tail}$, then $P(A)=0$ or 1.

Pf. $X_t = t B(\frac{1}{t})$. Then $\mathcal{F}_{tail}$ is mapped to $\mathcal{F}_0^+$.

Def. A process is called <u>adapted</u> to a filtration if X_t is $\mathcal{F}_t$ measurable $\forall t$.

Each process has a <u>natural filtration</u> - the smallest filtration that makes X_t $\mathcal{F}_t$ measurable.

<u>Adapted to 6-adic</u> $\iff$ X_n is constant on n-th generation 6-adic intervals.

Def. A stopping time wrt filtration $(\mathcal{F}_t)_{t\in I}$ is a function $T:\Omega\to I$: $\forall t\in I$ $\{\omega: T(\omega)\le t\}\in\mathcal{F}_t$.

Intuitively, by time t you know whether the event already happened. Already used 6-adic stopping times!

<u>Example</u>: <u>Hitting time</u> a) $I=\mathbb{N}$, (X_n) - an E-valued process, (E is anything!) $A\subset E$. Then $T=\min\{n: X_n\in A\}$ is a stopping time, since $\{T\le n\}=\bigcap_{k\le n}\{\omega: X_n(\omega)\in A\}\in\mathcal{F}_n$.

b) (X_t) has almost surely continuous trajectories, valued at metric space (E,ρ), $A\subset E$ - closed, $T=\inf\{t: X_t\in A\}$ - stopping time since $\{\omega: T(\omega)\le t\} = \bigcap_k \bigcap_{q\in\mathbb{Q}, q\le t} \{\rho(X_q, A)\le \frac{1}{k}\}\in\mathcal{F}_t$ (have to use rationals to avoid intersecting uncountably many sets).

For each stopping time, can define stopped σ-algebra $\mathcal{F}_T := \{A\in\mathcal{F}: \forall t\in I \ A\cap\{\omega: T(\omega)\le t\}\in\mathcal{F}_t\}$ - events that happens before stopped time T.

Thm (Strong Markov property of BM)

Let T be a stopping time wrt $(\mathcal{F}_t^+)$ - extended natural filtration for BM. Then $\{B(T+t)-B(T), t\ge 0\}$ is an SBM, independent of $\mathcal{F}_T^+$.

$\underbrace{B(T+t)-B(T)}_{=: B_t^{(T)}}$

Pf. Continuity is clear, so we need to establish normality and correlation. For this, take any $0\le t_1 < \dots < t_p$, and any continuous function of p variables, F, as well as $A\in\mathcal{F}_T^+$. Then

Claim. $E(F(B_{t_1}^{(T)}, \dots, B_{t_p}^{(T)})\cdot\chi_A) = E(F(B_{t_1},\dots,B_{t_p}))P(A)$.

The claim shows that $B_t^{(T)}$ has the same finite-dimension distributions as B_t and independent of A, thus proving the Theorem.

Pf of Claim Let $T_n := \min\{k2^{-n}: k2^{-n}\ge T\}$.

Then $T_n\to T$, T_n is a stopping time $\{T_n\le k2^{-n}\}\iff\{T\le k2^{-n}\}\in\mathcal{F}_{k2^{-n}}$, and, by dominated convergence,

$E(F(B^{(T)}_{t_1}, \dots B^{(T)}_{t_p})\chi_A) = \lim_{n\to\infty} E(F(B^{(T_n)}_{t_1}, \dots B^{(T_n)}_{t_p})\chi_A)$. So it is enough to prove the claim for T_n.

But $E(\dots) = \sum_{k=0}^{\infty} E(\chi_A \chi_{\{T_n = k2^{-n}\}} F(B_{k2^{-n}+t_1}, \dots, B_{k2^{-n}+t_p})) =$

$= \sum_{k=0}^{\infty} P(A \cap T = k2^{-n}) E(F(B_{t_1}, \dots B_{t_p})) = P(A) E(F(B_{t_1}, \dots, B_{t_p}))$ ■ $A \cap \{T_n = k2^{-n}\} \in \mathcal{F}^+_{k2^{-n}}$ independence + Markov

Reflection principle: a corollary:

T - stopping time. $B^*_t := \begin{cases} B_t, & t \le T \\ 2B_T - B_t, & t > T \end{cases}$ - reflected BM.

Then B^*_t - standard BM.

Pf B^*_t is continuous. By strong Markov, both $B(t+T) - B(T) =: B'(t)$ and $-B(t+T) + B(T) =: B^2(t)$ are SBM's independent of $B(s)$, $s \le T$. Thus $B_t = \begin{cases} B_t, & t \le T \\ B_T + B'_{t-T}, & t \ge T \end{cases}$ and $B^*_t = \begin{cases} B_t, & t \le T \\ B_T + B^2_{t-T}, & t \ge T \end{cases}$ have the same finite distributions. ■

Corollary. Let $M(t) := \max_{s \le t} B(s)$. Then $M(t) - B(t)$ has the same distribution as $|B(t)|$.

Pf. We'll prove a slightly weaker statement (just one-point agreement).

$P(M_t \ge a, B_t \le b) = P(B_t \ge 2a - b)$, if $b \le a$

Let $T_a = \inf\{t \ge 0: B_t = a\}$ - stopping time.

$P(M_t \ge a, B_t \le b) = P(T_a \le t, B_t \le b) = P(T_a \le t, B^{(T_a)}_{t-T_a} \le b - a) =$

since $B^{(T_a)}_{t-T_a} = B_t - B_{T_a} = B_t - a$

$P(T_a \le t, B^*_t \ge 2a - b) \underset{2a-b \ge a}{=} P(B^*_t \ge 2a - b) = P(B_t \ge 2a - b)$ ■

Let $R_B = \{t \le 1: B(t) = M(t)\}$ - the set of record times.